the squares of the three diagonals formed by connecting the extremities of said edges plus the square of the diagonal of the parallelepiped originating at the intersection of said three edges.

Whether the formula could be used in determining the properties of figures in space of four dimensions depends on whether the noncommutative character of vector multiplication does or does not affect the scalar part of the product of linear vector functions, in such space, in the particular case where such product is a homogeneous quadratic function of the vectors employed. If applicable, the number of vectors employed in the formula, and its form, would indicate its usefulness in such higher space.

507. Proposed by A. A. BENNETT, University of Texas.

With the use of the compasses alone construct a circle with area five times as great as that of a given circle. (This problem is said to be due to Napoleon I.)

I. Solution by I. L. Miller, Indiana University.

This problem is equivalent to the construction of $R\sqrt{5}$, where R is the radius of the given circle.

Let A be the center of the given circle. (Fig. 1) Take any point B on its circumference and construct a circle of radius equal to that of the given circle. Let this circle intersect the given circle in C

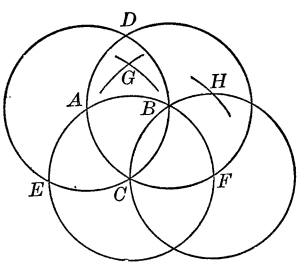

Fig. 1.

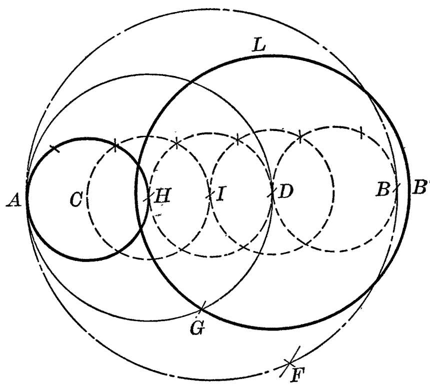

Fig. 2.

and D. Now with C as center construct another circle of radius R, intersecting the first two circles in E and F respectively. And finally with F as center construct another equal circle.

It is evident that $CD = R\sqrt{3}$. With E and F as centers and CD as radius strike arcs, intersecting in G. Then $CG = R\sqrt{2}$. With C as center and CG as radius strike an arc intersecting in H the circumference of the circle last drawn.

Then $EH = R\sqrt{5}$.

This also solves another problem due to Napoleon I; the quadrisection of the circumference of a circle by the use of the compasses alone, which is equivalent to the construction of $R\sqrt{2}$.

II. Solution by Gregory Breit, Student, Johns Hopkins University.

Let AC be the given circle, of radius r. (Fig. 2) From an arbitrary point A on its circumference, lay off three chords of radius length, reaching point H; HA is then a diameter of AC. With H as a center and radius r lay off another circle finding its diameter CI in the same manner as before. Continue this until five such circles are drawn with their centers in the line AB. With H as a center and $2r$ as a radius, lay off the circle AD. With I as center and $3r$ as radius lay off a circle AB. With a radius of length $5r$ cut this circle in F, using A as center. Describe an arc with radius BF from A as a center, cutting circle AD in G. With DG as radius and D as center describe the circle LGB' which is a circle whose area is five times the area of AC. For chord $AF = 5r$, and diameter $AB = 6r$. Hence, chord $BF = \sqrt{36r^2 - 25r^2} = r\sqrt{11}$, chord $AG =$ chord $BF = r\sqrt{11}$, and diameter $AD = 4r$.

Hence, $DG = \sqrt{16r^2 - 11r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

Also solved in various ways by J. W. BALDWIN, F. E. CANADAY, C. E. GITHENS, F. E. WOODS, J. E. HATCH, O. S. ADAMS, J. ROSENBAUM, J. M. STETSON, and J. E. MCMAHON, JR.

CALCULUS.

419. Proposed by C. C. YENN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

SOLUTION BY THE PROPOSER.

Using the formula $\int\int \{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2\}^{1/2} dydx$, we have, from the equation of the surface,

$$\partial z/\partial x = -z^{1/3}/x^{1/3}, \qquad \partial z/\partial y = -z^{1/3}/y^{1/3};$$

whence

$$S = 8\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} x^{-1/3}y^{-1/3}dydx. \tag{1}$$

To integrate with respect to y, let $y^2 = w^3$, so that $y^{-1/3}dy = \frac{3}{2}dw$, also put

$$A = (a^{2/3} - x^{2/3})x^{2/3}, \qquad B = (a^{2/3} - x^{2/3}). \tag{2}$$

Then

$$\int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} y^{-1/3}dy = \frac{3}{2}\int_0^B \{A + Bw - w^2\}^{1/2}dw$$

$$= \frac{3}{2}\left\{\frac{2w - B}{4}(A + Bw - w^2)^{1/2} + \frac{B^2 + 4A}{8}\sin^{-1}\left[\frac{2w - B}{(B^2 + 4A)^{1/2}}\right]\right\}_0^B.$$

Substituting the limits of integration, also the values of A and B from (2), we have from (1),

$$S = 6\int_0^a (a^{2/3} - x^{2/3})^{3/2}dx + 3\int_0^a (a^{2/3} - x^{2/3})(a^{2/3} + 3x^{2/3})x^{-1/3}\sin^{-1}\left\{\frac{a^{2/3} - x^{2/3}}{a^{2/3} + 3x^{2/3}}\right\}^{1/2} dx. \tag{3}$$

To evaluate the first integral, let $x = v^3$, $dx = 3v^2dv$, also put $a^{2/3} = \alpha^2$, then

$$6\int_0^a (a^{2/3} - x^{2/3})^{3/2}dx = 18\int_0^\alpha (\alpha^2 - v^2)^{3/2}v^2dv$$

$$= -18\frac{v(\alpha^2 - v^2)^{5/2}}{6}\Big|_0^\alpha + 3\alpha^2\int_0^\alpha (\alpha^2 - v^2)^{3/2}dv$$

$$= 3\alpha^2\left\{\frac{v(\alpha^2 - v^2)^{1/2}(5\alpha^2 - 2v^2)}{8} + \frac{3\alpha^4}{8}\sin^{-1}\left(\frac{v}{\alpha}\right)\right\}_0^\alpha$$

$$= \frac{9}{8}\alpha^6\frac{\pi}{2} = \frac{9\pi a^2}{16}.$$

To evaluate the second integral, put $x^{2/3} = \lambda$, $\frac{2}{3}x^{-1/3}dx = d\lambda$, also set $a^{2/3} = k$, then the second term of S in (3) becomes

$$\begin{aligned} &3 \times \frac{3}{2}\int_0^k (k - \lambda)(k + 3\lambda)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2} d\lambda \\ &= \frac{9}{2}(k^2\lambda + k\lambda^2 - \lambda^3)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}\Big|_0^k + \frac{9}{2}\int_0^k (k^2\lambda + k\lambda^2 - \lambda^3)\frac{kd\lambda}{(k + 3\lambda)\{\lambda(k - \lambda)\}^{1/2}}, \end{aligned} \tag{4}$$

where the first term vanishes at both limits of integration. To integrate the second term, let $\sqrt{\lambda(k - \lambda)} = \lambda\xi$, then $\lambda = k/(1 + \xi^2)$, $d\lambda = -2k\xi d\xi/(1 + \xi^2)^2$; and the second term of (4) becomes

$$-9k^3\int_\infty^0 \left(\frac{1}{1 + \xi^2} + \frac{1}{(1 + \xi^2)^2} - \frac{1}{(1 + \xi^2)^3}\right)\frac{d\xi}{\xi^2 + 4} = 9k^3 \lim_{c=\infty}\int_0^c \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)}.$$